

Towards a complete theory of fermion masses and mixings with $SO(3)$ Family Symmetry and 5d $SO(10)$ unification

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ABSTRACT: We construct a complete 4d model of fermion masses and mixings in the Pati-Salam $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ framework governed by an $SO(3)$ gauged Family Symmetry. The relevant low energy effective Yukawa operators are constructed so that the $SO(3)$ flavons enter at the simplest possible one-flavon level, with couplings enforced by an additional $U(1) \times Z_2$ symmetry. The simplicity of the flavon sector allows the messenger sector to be fully specified, allowing the ultraviolet completion of the model at the 4d renormalizable level. The model predicts approximate tri-bimaximal lepton mixing via the see-saw mechanism with sequential dominance, and vacuum alignment of flavons, with calculable deviations described by the neutrino sum rule. We perform a numerical analysis of the emerging charged fermion spectra and mixings. The 4d model is shown to result from a 5d orbifold GUT model based on $SO(3) \times SO(10)$, where small flavon vacuum expectation values (VEVs) originate from bulk volume suppression.

KEYWORDS: Beyond Standard Model, Quark Masses and SM Parameters.

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1. Introduction

The pattern of charged fermion masses and quark mixing angles is described by 13 parameters in the minimal Standard Model (3 charged lepton masses, 6 quark masses, 3 quark mixing angles and 1 quark CP violating phase). The discovery of neutrino masses and lepton mixing angles requires a further 9 parameters if neutrinos are Majorana (3 neutrino masses, 3 lepton mixing angles and 3 lepton CP violating phases), or 7 parameters if neutrinos are Dirac (2 fewer lepton phases). The discovery of neutrino mass has consequently increased the number of parameters in the flavour sector dramatically, providing further motivation to understand the pattern of fermion masses and mixings. The neutrino sector also provides additional data and clues which could enable this goal to be achieved. For example, the neutrino sector is distinguished by having two large mixing angles, one small mixing angle, and very small mass eigenvalues [1]. Recently, as neutrino data has become more accurate, intriguing patterns of mixing angles in the lepton sector have begun to emerge [2].

A particularly promising strategy, leading to an effective reduction of the number of independent Yukawa couplings, is the idea of Grand Unified Theories (GUTs) [3–5]. GUTs allow one to understand the variety of Standard Model (SM) fermions as arising from a few fundamental representations and often lead to strong correlations amongst the corresponding low energy Yukawa couplings. A typical example could be the relation $m_b \approx m_\tau$ [3] emerging in a natural way in the class of GUT models based on $SU(5)$ gauge symmetry [3]. Another example would be the interesting correlation between $b - \tau$ unification and the large atmospheric mixing angle [6, 7] revealed in a class of renormalizable $SO(10)$ models with type-II contribution dominating the seesaw formula [8–15]. Although the SM flavour problem is just one of many questions confronting GUTs (and typically is not the main motivation to go beyond the SM gauge structure) it is still very interesting that the detailed information on the quark and lepton masses and mixing patterns can lead to severe constraints on model building. For example it has been shown that minimal renormalizable SUSY $SO(10)$ does not provide a consistent description of the recent fermion mass and mixing data [16–24]. In general GUT models do not provide a full understanding of the observed pattern of quark and lepton masses and mixings. In particular, without other assumptions there is usually no explanation of the observed quark and lepton mass hierarchies, spanning the three families. Although the question of the fermion mass hierarchy has been qualitatively addressed in approaches based on higher dimensional orbifold GUTs (c.f. for instance [25–32] and references therein) in which the smallness of the first and second family masses are traced back to their location in the extra dimensions, see for instance [31, 33–38], these constructions do not provide any understanding of the origin of the small Cabibbo-Kobayashi-Maskawa (CKM) quark sector mixing together with the observed bi-large mixing in the lepton sector. Some GUT models also include extra (global or local) Family Symmetry, spontaneously broken by flavon VEVs, in order to provide a more predictive framework [39]. The extra Family Symmetry acting nontrivially among different flavours across the SM matter families provides additional constraints on the Yukawa textures. The problem of quark and lepton mixing patterns has been recently addressed in models based on extra continuous Family Symmetries like $U(1)$ [40–45], $SU(2)$ [46, 47], $SU(3)$ [48, 49], $SO(3)$ [50, 51] or discrete subgroups of continuous symmetries like for instance D_5 [52], D_4 [53], D_3 [54, 55], S_4 [56], A_4 [57].

One of the challenges facing such GUT and Family Symmetry models is to provide a convincing explanation of the observed (approximate) tri-bimaximal lepton mixing, corresponding to a maximal atmospheric angle $\tan \theta_{23} \approx 1$, a zero reactor angle $\theta_{13} \approx 0$ and a solar angle $\sin \theta_{12} \approx 1/\sqrt{3}$ [58]. The possibility that the tri-bimaximal neutrino mixing matrix involves square roots of simple ratios motivates models in which the mixing angles are independent of the mass eigenvalues. One such class of models are see-saw models with sequential dominance of right-handed neutrinos [59–62]. In sequential dominance, a neutrino mass hierarchy is shown to result from having one of the right-handed neutrinos give the dominant contribution to the see-saw mechanism, while a second right-handed neutrino gives the leading sub-dominant contribution, leading to a neutrino mass matrix with naturally small determinant. In a basis where the right-handed neutrino mass matrix and the charged lepton mass matrix are diagonal, the atmospheric and solar neutrino

mixing angles are determined in terms of ratios of Yukawa couplings involving the dominant and subdominant right-handed neutrinos, respectively. If these Yukawa couplings are simply related in some way, then it is possible for simple neutrino mixing angle relations, such as appear in tri-bimaximal neutrino mixing, to emerge in a simple and natural way, independently of the neutrino mass eigenvalues. Specifically, if the dominant right-handed neutrino couples equally to the second and third family, with a zero coupling to the first family, then this will result in a maximal atmospheric mixing angle $\tan \theta_{23} \approx 1$. If the leading subdominant right-handed neutrino couples equally to all three families, and if these couplings are orthogonal to the couplings of the dominant right-handed neutrino, then this will result in a tri-bimaximal solar neutrino mixing angle $\sin \theta_{12} \approx 1/\sqrt{3}$, and a zero reactor angle $\theta_{13} \approx 0$, assuming that the third right-handed neutrino is completely decoupled from the see-saw mechanism. This is called constrained sequential dominance (CSD) [63]. In realistic models there will be corrections to tri-bimaximal mixing from charged lepton mixing, resulting in testable predictions and sum rules for lepton mixing angles [63, 64].

In order to achieve the CSD Yukawa relations it seems to be necessary to introduce a non-Abelian horizontal Family Symmetry spanning the three families. The Family Symmetry is then broken by flavons, and effective Yukawa operators may be constructed where the aligned flavon VEVs provide the required CSD relations between the Yukawa couplings. This strategy has been followed for models based on the Family Symmetry $SO(3)$ [63], $SU(3)$ [65], or their discrete subgroups [66, 67] (see also [68]). The choice of $SU(3)$, or a discrete subgroup of it such as $\Delta(27)$ [67], has the advantage that it enables both the left and right handed chiral fermions to both transform under the Family Symmetry as triplets, permitting unification into a single $SU(3) \times SO(10)$ multiplet $(3, 16)$. The lowest order effective Yukawa operators for $SU(3)$ must then involve a minimum of two anti-triplet flavon insertions. The choice of $SO(3)$, or a discrete subgroup of it such as A_4 [66], requires that only one type of chiral fermion transform under the Family Symmetry, while the other type is a family singlet, in order to avoid trivial Family Symmetry contractions. The disadvantage is that it seems to not allow a similar unification into a single $SO(3) \times SO(10)$ multiplet $(3, 16)$, however it has the advantage that the lowest order effective Yukawa operators for $SO(3)$ involve only one triplet flavon insertion, and are therefore simpler. However, in practice, this advantage has so far not been exploited since the lowest order effective operator has only been used for the largest Yukawa coupling associated with the third family, while the operators associated with the first and second family were assumed to involve three flavons, in order to account for the required suppression, where the resulting scheme required large additional symmetries [63].

In the present paper we shall exploit the simplicity of $SO(3)$ by assigning to all three families effective Yukawa operators involving just single flavon insertions in the Dirac sector. We assume that there are essentially two types of flavons, one type which develops large VEV and may be used to describe the third family, and a second type which develops a small VEV and will be assigned to the lighter families. In our approach the main role of the horizontal symmetry (realized at the quantum level in terms of extra interactions of matter with relevant flavon fields) is to explain the *correlations* among the Yukawa entries rather than their *hierarchy*, which is accounted for by the small flavon VEVs. Since the flavons

enter the effective operators at the lowest possible level, this allows us to reduce the usually cumbersome extra symmetries considerably, as there is no need to suppress wide classes of effective operators up to high order in the number of flavon insertions. The resulting relative simplicity of the flavon sector encourages us to go beyond the effective non-renormalizable Yukawa operator description in [63, 65–67] and construct explicitly the renormalizable messenger sector, leading to an ultraviolet completion of the model. The messenger sector allows for effectively different expansion parameters in the different charged sectors, and the effective Yukawa and Majorana matrices are then constructed. We perform a numerical analysis which shows that the model provides an excellent fit to the charged fermion mass spectrum. The model also predicts approximate tri-bimaximal lepton mixing via CSD due to vacuum alignment of flavon VEVs, with calculable deviations described by the neutrino sum rule. The strong hierarchy in the charged fermion sector gets cancelled in the neutrino sector, via the see-saw mechanism with sequential dominance, leading to $m_2 \sim m_3$ for the lowest order neutrino masses, with the mild neutrino hierarchy $m_2/m_3 \sim 1/5$ produced by higher order corrections necessarily present in the model. Finally we show that the 4d model here can result from a 5d orbifold GUT model based on $\text{SO}(3) \times \text{SO}(10)$, leading to a full $\text{SO}(10)$ unification of the $\text{SO}(3)$ model, and an explanation of the small flavon VEV responsible for the fermion mass hierarchy in terms of bulk volume suppression. The synthesis of non-Abelian Family Symmetry with orbifold GUTs provides an attractive way of simplifying the Yukawa operators required by explaining the fermion mass hierarchy in terms of a single suppressed flavon VEV rather than a higher order operator. Such a simplification of the Yukawa operators at the non-renormalizable level is instrumental in allowing us to provide the ultraviolet completion of the model in terms of an explicit messenger sector.

The layout of the remainder of the paper is as follows. In section II we discuss the 4d model at the effective operator level, specifying the symmetry and field content of the model, and performing a full operator analysis of the effective Dirac and Majorana operators. We show that it leads to approximate tri-bimaximal lepton mixing and a normal neutrino mass hierarchy. In section III we discuss the complete 4d model including the messenger sector responsible for the effective operators. In section IV we perform a numerical analysis of the model, where we show that the parameters of the model can provide a successful fit for the quark masses and mixings using the charged lepton masses as inputs. In section V we discuss the embedding of the model into a 5d $\text{SO}(3) \times \text{SO}(10)$ orbifold GUT model, in which small flavon VEVs can be accounted for by volume suppression, and the full $\text{SO}(10)$ unification of the model is manifested. section VI concludes the paper.

2. The 4d effective non-renormalizable model

2.1 The symmetry

We work in a class of supersymmetric Pati-Salam models based on the gauge symmetry group $\text{SU}(4) \otimes \text{SU}(2)_L \otimes \text{SU}(2)_R$, which is supposed to be spontaneously broken at some high scale M_G (typically $\gtrsim 10^{15}$ GeV) to the ordinary $\text{SU}(3)_c \otimes \text{SU}(2)_L \otimes \text{U}(1)_Y$ of the minimal supersymmetric standard model (MSSM). Though this is not a fully unified description of

both the strong and electroweak interactions (just as the Standard Model does not fully unify the electromagnetic and weak nuclear force), this structure is liberal enough to let the left-handed matter fields transform nontrivially under the horizontal $SO(3)$ unlike the right-handed SM fermions (that are $SO(3)$ singlets,¹) while still rigid enough to give rise to a set of nontrivial correlations among the quark and lepton Yukawa couplings. The horizontal $SO(3)$ is a gauged Family Symmetry while the extra $U(1) \otimes Z_2$ factors are supposed to be approximate² global symmetries of the model at the Pati-Salam level. We expect these symmetries to be broken spontaneously by the VEVs of a set of flavon fields transforming trivially under the gauge symmetry.³

2.2 The field content

We assume the minimal Pati-Salam matter content and let the left-handed matter fermions (transforming like $(4, 2, 1)$ under $SU(4) \otimes SU(2)_L \otimes SU(2)_R$) denoted by \vec{F} form a triplet under the $SO(3)$ flavour symmetry, while the right-handed components F_1^c , F_2^c and F_3^c (behaving like $(\bar{4}, 1, 2)$ under the PS symmetry) are supposed to be $SO(3)$ singlets. There is one copy of the Higgs bidoublet driving the SM spontaneous symmetry breakdown and a pair of Higgs fields (denoted by $H \oplus \bar{H}$ and $H' \oplus \bar{H}'$) responsible for the proper breaking of the PS symmetry and the Majorana masses of neutrinos. Last, there is an extra Higgs field Σ transforming as $(15, 1, 3)$ of Pati-Salam symmetry that gives rise to the desired Georgi-Jarlskog [69] Clebsch factors in the charged matter sector while keeping the effective neutrino Dirac Yukawa couplings intact.

Concerning the flavon sector, we follow the generic construction in [63]. Since the model will involve a minimal number of flavon insertions, their extra charges are chosen to be opposite to those of F_1^c , F_2^c and F_3^c so that a particular flavon is associated with a particular column of the Yukawa matrix, at lowest order. The full set of the effective theory matter, Higgs and flavon fields and their transformation properties are given in table 1.

With all this information at hand, one can easily infer the shape of the lowest level effective operators allowed by the gauge and extra symmetries. Let us start with the Dirac Yukawa texture, that exhibit the effects of the $SO(3)$ horizontal symmetry in its full glory, in particular in the neutrino sector.

2.3 The Dirac sector

The main feature of our construction is the simplicity of the leading effective operators (responsible in particular for the desired tri-bimaximal structure of the neutrino Dirac and Majorana Yukawa matrices, c.f. section 2.4), in particular the fact that they all emerge

¹This choice leads to the typical correlations among entries in columns of the relevant Dirac Yukawa matrices (in LR notation), just in a way it is expected in a class of lepton sector models with sequential dominance.

²This is there namely to avoid problems with Goldstone bosons and/or topological defects below the extra symmetry breakdown scale.

³Thus keeping the GUT-like gauge coupling convergence intact. (However, there is no need to demand an exact $SU(5)$ or $SO(10)$ -like gauge coupling unification here as $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ is not a simple group.)

field	$SU(4) \otimes SU(2)_L \otimes SU(2)_R$	$SO(3)$	$U(1)$	Z_2
\vec{F}	$(4, 2, 1)$	3	0	+
F_1^c	$(\bar{4}, 1, 2)$	1	+2	-
F_2^c	$(\bar{4}, 1, 2)$	1	+1	+
F_3^c	$(\bar{4}, 1, 2)$	1	-3	-
h	$(1, 2, 2)$	1	0	+
H, \bar{H}	$(4, 1, 2), (\bar{4}, 1, 2)$	1	± 3	+
H', \bar{H}'	$(4, 1, 2), (\bar{4}, 1, 2)$	1	∓ 3	+
Σ	$(15, 1, 3)$	1	-1	-
$\vec{\phi}_3$	$(1, 1, 1)$	3	+3	-
$\vec{\phi}_{23}$	$(1, 1, 1)$	3	-2	-
$\vec{\phi}_{123}$	$(1, 1, 1)$	3	-1	+
$\vec{\phi}_{12}$	$(1, 1, 1)$	3	0	+
$\vec{\phi}_{23}$	$(1, 1, 1)$	3	0	-

Table 1: The basic Higgs, matter and flavon content of the model.

at one flavon insertion level, with all the advantages over the former constructions (the simplicity of the extra symmetries and the would-be Froggatt-Nielsen messenger sector):

$$W_Y^{\text{leading}} = \frac{1}{M} y_{23} \vec{F} \cdot \vec{\phi}_{23} F_1^c h + \frac{1}{M} y_{123} \vec{F} \cdot \vec{\phi}_{123} F_2^c h + \frac{1}{M} y_3 \vec{F} \cdot \vec{\phi}_3 F_3^c h + \dots \quad (2.1)$$

where M stands for the masses of the relevant Froggatt-Nielsen messenger fields while the ellipsis covers the subleading terms necessary for the proper description of the quark sector details (hierarchies and the CKM mixing parameters, the first generation masses etc.):

$$W_Y^{\text{subl.}} = \frac{1}{M^2} y_{GJ} \vec{F} \cdot \vec{\phi}_{23} F_2^c \Sigma h + \frac{1}{M^2} y_{12} \vec{F} \cdot (\vec{\phi}_3 \times \vec{\phi}_{12}) F_3^c h + \frac{1}{M^3} \tilde{y}_{23} \vec{F} \cdot \vec{\phi}_{23} (\vec{\phi}_3 \cdot \vec{\phi}_{23}) F_3^c h + \dots \quad (2.2)$$

As we claimed before, we exploit the $SO(3)$ horizontal symmetry to understand the correlations among the various Yukawa entries rather than their exact hierarchy.⁴ Instead, we equip some of the flavon and Higgs VEVs with extra suppression factors (with respect to their natural values dictated by the relevant symmetry breaking scales) and let these factors tell their favorite values just upon fitting all the quark and lepton data. Remarkably enough, there is an option to connect all of them to just one universal suppression scale, that could find a natural justification for instance in higher-dimensional constructions.

As an example, consider the CSD structure of the neutrino Dirac Yukawa matrix Y^ν emerging from eqs. (2.1)–(2.2) (for further details see also formula (2.7)). The lepton mixing data do not specify the overall magnitudes of its first and second columns, only

⁴As a matter of fact, such attempts are questionable indeed, because to understand hierarchies an extra symmetry does not help much unless a scale at which it becomes broken is specified, i.e. there is always an extra ingredient needed to accomplish such a goal.

the correlations among their entries. This is precisely where the horizontal symmetry is supposed to play an important role. Only after its embedding into a (partially) unified framework like the Pati-Salam gauge model, their correlations with the quark sector Dirac Yukawa couplings trigger the need of a particular suppression of the first and second column entries with respect to the third column ones.

Thus, it is quite natural to let these two requirements of intrinsically different origins be justified from two different sources like we propose here – the SO(3) symmetry shall govern the rescaling-invariant quantities like the lepton mixing angles (in seesaw-type models) while the charged matter sector hierarchies emerge from the suppression of the flavon VEVs $\langle\phi_{23}\rangle$, $\langle\phi_{123}\rangle$ and $\langle\phi_{12}\rangle$ (and similarly for H' Higgs field). We let these VEVs (driven by naturalness to roughly the same order of magnitude corresponding to the scale of the SO(3) (and Pati-Salam) symmetry breaking, at least if it is one-step) be suppressed by extra factors called δ_{23} , δ_{123} and δ_{12} with respect to the VEV of ϕ_3 , namely:

$$|\langle\phi_{123}\rangle| \sim \delta_{123}|\langle\phi_3\rangle|, \quad |\langle\phi_{23}\rangle| \sim \delta_{23}|\langle\phi_3\rangle| \quad \text{and} \quad |\langle\phi_{12}\rangle| \sim \delta_{12}|\langle\phi_3\rangle| \quad (2.3)$$

As we shall see later in section 5 all these suppression factors can be justified in terms of a universal suppression factor δ coming from extra dimensional dynamics. Note that we keep the VEV of the ‘‘Georgi-Jarlskog’’ flavon $\tilde{\phi}_{23}$ at the natural scale $|\langle\tilde{\phi}_{23}\rangle| \sim |\langle\phi_3\rangle|$ and let the slight suppression of the second generation masses come from the higher level nature of the relevant effective operator in eq. (2.2).

In a similar manner, the Majorana sector structure shall be affected by the requirement of having the VEV of H' well below the VEV of H , namely

$$|\langle H'\rangle| \sim \delta_H|\langle H\rangle|. \quad (2.4)$$

The alignment of the flavon VEVs will be assumed to be given by [63]:

$$\langle\phi_3\rangle^T \propto (0, 0, 1), \langle\phi_{23}\rangle^T \propto (0, 1, -1), \langle\tilde{\phi}_{23}\rangle^T \propto (0, 1, 1), \langle\phi_{123}\rangle^T \propto (1, 1, 1), \langle\phi_{12}\rangle^T \propto (1, 1, 0) \quad (2.5)$$

and will not be discussed further here.

After the appropriate flavour symmetry breaking this structure gives rise to the Dirac mass matrices like (in an obvious symbolic LR notation):

$$Y_{LR}^f = \begin{pmatrix} 0 & y_{123}\varepsilon_{123}^f & y_{12}\varepsilon_{12}^f\varepsilon_3^f \\ y_{23}\varepsilon_{23}^f & y_{123}\varepsilon_{123}^f + C^f y_{GJ}\tilde{\varepsilon}_{23}^f\sigma & \tilde{y}_{23}(\tilde{\varepsilon}_{23})^2\varepsilon_3 \\ -y_{23}\varepsilon_{23}^f & y_{123}\varepsilon_{123}^f + C^f y_{GJ}\tilde{\varepsilon}_{23}^f\sigma & y_3\varepsilon_3^f \end{pmatrix} \quad (2.6)$$

where $C^f = -2, 0, 1, 3$ for $f = u, \nu, d, e$ are the traditional Clebsch-Gordon coefficients responsible for the distinct charged sector hierarchies, σ denotes the (normalized) VEV of the Georgi-Jarlskog field $\sigma \equiv \langle\Sigma\rangle/M_f$ and ε_x^f stands for the various flavon VEV factors $\langle\phi_x\rangle/M_f$.

Note that the choice of the Σ field giving rise to the desired Georgi-Jarlskog Clebsch factor in the charged sector Yukawas $Y^{e,u,d}$ is practically unique. Since we need to preserve the tight CSD homogeneity of the second column of the neutrino dirac mass matrix, the

effect of the Σ VEV should be strongly suppressed in Y^ν by the relevant Clebsch factor of Σ VEV in the neutrino direction. Then, Σ transforming like $(15, 1, 3)$ under $SU(4)_C \otimes SU(2)_R \otimes SU(2)_L$ is the simplest choice that can satisfy this requirement.

2.4 The Majorana neutrino sector

Assuming the vacuum alignment as in the previous subsection, the neutrino Yukawa matrix takes the form:

$$Y_{LR}^\nu = \begin{pmatrix} 0 & y_{123}\varepsilon_{123}^\nu & y_{12}\varepsilon_{12}^\nu\varepsilon_3^\nu \\ y_{23}\varepsilon_{23}^\nu & y_{123}\varepsilon_{123}^\nu & \tilde{y}_{23}(\varepsilon_{23}^\nu)^2\varepsilon_3^\nu \\ -y_{23}\varepsilon_{23}^\nu & y_{123}\varepsilon_{123}^\nu & y_3'\varepsilon_3^1 + y_3\varepsilon_3^\nu \end{pmatrix} \quad (2.7)$$

Assuming that the right-handed neutrino associated with the first column gives the dominant contribution to the see-saw mechanism, the second right-handed neutrino gives the leading subdominant contribution, and the third column gives the smallest contribution, then this form of neutrino Yukawa matrix corresponds to constrained sequential dominance (CSD), and will lead to tri-bimaximal lepton mixing as discussed in [63].

The Majorana right-handed neutrino mass matrix must be approximately diagonal, so as not to lead to significant corrections to the Yukawa matrix in the diagonal right-handed neutrino mass basis, and in addition it must be sufficiently hierarchical to ensure that the right-handed neutrinos dominate sequentially as described above. The (leading order) structure of the neutrino Majorana mass matrix is triggered by the choice of the $U(1) \times Z_2$ charges of the heavy Higgs fields H and H' . Assuming the hierarchy among the VEVs of the flavon and Higgs fields given by eqs. (2.3)–(2.4) the lowest level effective operators allowed by the extra symmetries are:

$$W_M^{lead.} = \frac{1}{M_\nu^3} w_1 F_1^{c2} H H' \phi_{23}^2 + \frac{1}{M_\nu^3} w_2 F_2^{c2} H H' \phi_{123}^2 + \frac{1}{M_\nu} w_3 F_3^{c2} H^2 + \dots \quad (2.8)$$

Assuming the relevant messengers to be the same as for the Dirac neutrino sector (see later) these terms generate a diagonal Majorana mass matrix

$$M_{RR}^\nu = \text{diag}(w_1\varepsilon_{23}^{\nu^2}\delta_H, w_2\varepsilon_{123}^{\nu^2}\delta_H, w_3)M_3$$

where as before ε_x^ν denotes $\langle\phi_x\rangle/M$. The see-saw formula $m_\nu = Y_{LR}^\nu M_{RR}^\nu^{-1} Y_{LR}^{\nu T} v^2$ leads to three contributions to the light neutrino mass matrix, from each of the three right-handed neutrinos, the first and second of order $\delta_H^{-1}v^2/M_3$, and the third of order $\varepsilon_3^{\nu^2}v^2/M_3$. With sufficiently small δ_H the third right-handed neutrino becomes decoupled and irrelevant for the see-saw mechanism. Such a simple Majorana structure has several noteworthy features. In particular, the would-be δ -suppressions associated to the ϕ_{23} and ϕ_{123} VEVs entering the Dirac neutrino Yukawa through the leading operators given in eq. (2.1) is cancelled in the seesaw formula by the suppression factors present in M_{RR}^ν . The ε_x^ν suppression factors similarly cancel, leading to $m_2 \sim m_3$, in contrast to the strong hierarchy in the charged matter spectra. Note, however, that $m_1 \ll m_2$.

The above see-saw cancellations, though welcome from the point of view of making the hierarchy between m_2 and m_3 mild, are apparently too efficient and at leading order

lead to no hierarchy at all, $m_2 \sim m_3$. However, apart from the leading terms given above, the extra symmetries allow for many subleading terms, for instance:⁵

$$\begin{aligned}
 W_M^{subl.} = & \frac{w_4}{M^4} F_1^{c2} H' H' (\vec{\phi}_3 \times \vec{\phi}_{123}) \cdot \vec{\phi}_{23} + \frac{w_5}{M^4} F_2^{c2} H H' (\vec{\phi}_{23} \times \vec{\phi}_{12}) \cdot \vec{\phi}_{23} + \\
 & + \frac{w_6}{M^5} F_2^{c2} H' H' (\vec{\phi}_3 \cdot \vec{\phi}_{23}) (\vec{\phi}_3 \cdot \vec{\phi}_{23}) + \frac{w_7}{M^4} F_1^c F_2^c H H' (\vec{\phi}_{23} \times \vec{\phi}_{123}) \cdot \vec{\phi}_{12} + \\
 & + \frac{w_8}{M^5} F_1^c F_2^c H' H' (\vec{\phi}_3 \cdot \vec{\phi}_{23}) (\vec{\phi}_{12} \cdot \vec{\phi}_{23}) + \frac{w_9}{M^4} F_1^c F_3^c H H' (\vec{\phi}_3 \times \vec{\phi}_{23}) \cdot \vec{\phi}_{12} + \\
 & + \frac{w_{10}}{M^6} F_2^c F_3^c H H (\vec{\phi}_{123} \times \vec{\phi}_{23}) \cdot \vec{\phi}_{23} (\vec{\phi}_{23} \cdot \vec{\phi}_{123}) + \frac{w_{11}}{M^6} F_2^c F_3^c H H' (\vec{\phi}_3 \times \vec{\phi}_{23}) \cdot \vec{\phi}_{23} (\vec{\phi}_{23} \cdot \vec{\phi}_3) + \dots
 \end{aligned} \tag{2.9}$$

With this information at hand the structure of the Majorana mass matrix reads⁶

$$M_{RR}^\nu = \begin{pmatrix} \mathcal{O}(\varepsilon_{23}^{\nu 2} \delta_H, \varepsilon_{123}^\nu \tilde{\varepsilon}_{23}^\nu \varepsilon_3^\nu \delta_H^2) & \mathcal{O}(\varepsilon_{123}^\nu \varepsilon_{23}^\nu \varepsilon_{12}^\nu \delta_H, \varepsilon_{12}^\nu \varepsilon_3^\nu \tilde{\varepsilon}_{23}^{\nu 2} \delta_H^2) & \mathcal{O}(\varepsilon_3^\nu \varepsilon_{23}^\nu \varepsilon_{12}^\nu \delta_H) \\ \cdot & \mathcal{O}(\varepsilon_{123}^{\nu 2} \delta_H, \varepsilon_{12}^\nu \varepsilon_{23}^\nu \tilde{\varepsilon}_{23}^\nu \delta_H, \dots) & \mathcal{O}(\varepsilon_{123}^{\nu 2} \varepsilon_{23}^\nu \tilde{\varepsilon}_{23}^{\nu 2}, \dots) \\ \cdot & \cdot & \mathcal{O}(1) \end{pmatrix} \frac{\langle H \rangle^2}{M} \tag{2.10}$$

Assuming $\delta_{123} \sim \delta_{23} \sim \delta_{12} \sim \delta_H \equiv \delta$ (leading to $\varepsilon_{123}^\nu \sim \varepsilon_{23}^\nu \sim \varepsilon_{12}^\nu \sim \delta \varepsilon_3^\nu \sim \delta \tilde{\varepsilon}_{23}^\nu$, c.f. formula (2.3)) the lepton mixing angles emerging from the Majorana sector are:

$$\theta_{12}^{RR} \sim \mathcal{O}(\tilde{\varepsilon}_{23}^{\nu 2}), \quad \theta_{13}^{RR} \sim \mathcal{O}(\delta^3 \varepsilon_3^\nu), \quad \theta_{23}^{RR} \sim \mathcal{O}(\delta^3 \varepsilon_3^\nu \tilde{\varepsilon}_{23}^\nu) \tag{2.11}$$

Since $\varepsilon_3^\nu \sim \tilde{\varepsilon}_{23}^\nu \ll 1$ (see section 3.4) we find that these angles are small enough not to disturb the required CSD Dirac sector correlations significantly, which would spoil the tri-bimaximal prediction. However the beneficial consequence of the effective operators is that one has enough room to smear the unwanted degeneracy of the first and second heavy Majorana masses restoring the validity of the second CSD hierarchy condition [63], leading to $m_2/m_3 \sim 1/5$ by a suitable choice of parameters.

3. The 4d renormalizable model

3.1 The messenger sector

We now present a renormalizable 4d theory which gives rise to the effective nonrenormalizable operators of the previous section. The effective non-renormalizable operators will arise from the exchange of heavy messenger fields. In this subsection we shall describe the messenger sector responsible for the effective Dirac operators. Note that the construction of the full model at the renormalizable level is greatly facilitated by the simplicity of the model at the effective operator level, in particular the fact that the simplest operators correspond to the insertion of only one flavon.

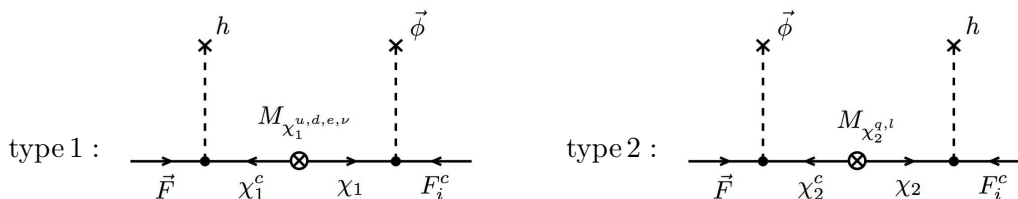
At the level of one flavon insertion operators dominating the Dirac Yukawa structures there are in principle two distinct classes of Froggatt-Nielsen operators behind, namely:

⁵Here we typically omit the allowed SO(3) contractions that drop out in the mass matrix because of the orthogonality of the relevant flavon VEVs, for instance $\frac{1}{M^5} F_2^{c2} H' H' \vec{\phi}_3 (\vec{\phi}_{23} \cdot \vec{\phi}_{23})$, $\frac{1}{M^5} F_1^c F_2^c H' H' \vec{\phi}_3 \cdot \vec{\phi}_{12}$ etc.

⁶Only the leading contributions (in number of suppressions in δ_H and ε_{23}^ν , ε_{123}^ν and ε_{12}^ν) are displayed.

field	$SU(4) \otimes SU(2)_L \otimes SU(2)_R$	$SO(3)$	$U(1)$	Z_2
χ_1, χ_1^c	$(4, 1, 2), (\bar{4}, 1, 2)$	3	0	+
$\tilde{\chi}_1, \tilde{\chi}_1^c$	$(4, 1, 2), (\bar{4}, 1, 2)$	1	0	-
χ_2, χ_2^c	$(4, 2, 1), (\bar{4}, 2, 1)$	1	± 3	-
$\psi, \bar{\psi}$	$(\bar{10}, 1, 3), (10, 1, 3)$	1	0	+
Ψ	$(1, 1, 1)$	1	0	-

Table 2: The “level-1” messenger sector of the model responsible for the desired Dirac Yukawa structures.



The main difference between them stems from the position of the flavon and MSSM-like Higgs doublet insertions determining the transformation properties of the relevant messenger fields χ_1 and χ_2 . The Pati-Salam quantum numbers of χ_1 are $(4, 1, 2)$ while χ_2 transforms as $(4, 2, 1)$. The χ_1 messenger is “universal” as it feels only the quantum numbers of \vec{F} and h while χ_2 is “flavon specific”, because it carries the flavon extra charges (and thus should be called χ_2^ϕ).

Upon the Pati-Salam spontaneous symmetry breaking the χ_1 multiplet splits into four distinct states (denoted in what follows by superscripts u, d, e, ν) while χ_2^ϕ decays into just 2 states because it does not feel the $SU(2)_L$ charges of the quarks and leptons (that is why it is furnished by only a pair of superscripts q, l). Playing with masses of $\chi_1^{u,d,e,\nu}$ one affects uniformly all the linear terms within a specific Yukawa matrix, while adjusting the masses of $\chi_2^{\phi;q,l}$ leads to changes of entries generated by the appropriate flavons (the specific χ_2^ϕ is associated to), but without differences between those in the up and down and charged lepton and neutrino sectors respectively.

Remarkably enough, the minimal set of messengers leading to potentially realistic quark and lepton Dirac Yukawa textures is very concise, as shown in table 2.

The interactions of the relevant messengers with the matter, flavon and Higgs fields are given by

$$\begin{aligned}
 W_\chi = & \vec{\chi}_1 \cdot (y_{\chi_1 F_1^c \phi_{23}} F_1^c \vec{\phi}_{23} + y_{\chi_1 F_2^c \phi_{123}} F_2^c \vec{\phi}_{123} + y_{\chi_1 F_3^c \phi_3} F_3^c \vec{\phi}_3) + y_{\chi_1^c F h} \vec{\chi}_1^c h + \quad (3.1) \\
 & + y_{\chi_1 \tilde{\chi}_1^c \phi_{12}} (\vec{\chi}_1 \times \vec{\chi}_1^c) \cdot \vec{\phi}_{12} + y_{\chi_2 F_3^c h} \chi_2 F_3^c h + y_{\chi_2^c F \phi_3} \vec{F} \cdot \vec{\phi}_3 \chi_2^c + y_{\tilde{\chi}_1 \chi_1^c \tilde{\phi}_{23}} \tilde{\chi}_1 (\vec{\chi}_1^c \cdot \vec{\phi}_{23}) + y_{\tilde{\chi}_1 F_2^c \Sigma} \tilde{\chi}_1 F_2^c \Sigma
 \end{aligned}$$

The internal structure of the lowest level Dirac operators is depicted in figure 1. It is assumed that there is only one light enough flavon specific messenger of type 2 (c.f. discussion below formula (3.1)) associated with $\vec{\phi}_3$.

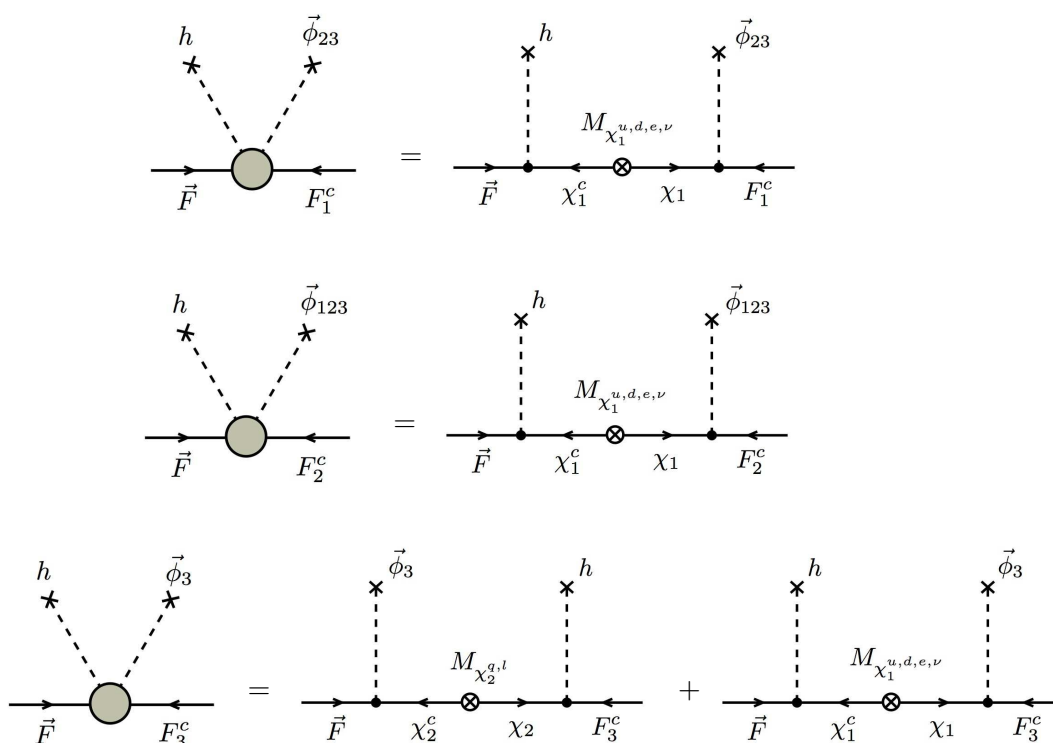


Figure 1: The structure of a typical contributions to the Dirac masses of matter fermions. Since the χ_1 messenger does not “feel” the extra quantum numbers of F_x^c and $\vec{\phi}_y$ these topologies are generic for all the Dirac entries. On the other hand, the $U(1) \times Z_2$ charges of the χ_2 messenger dominating the 33 entries are such that it can (at the lowest level) accompany only the $\vec{\phi}_3$ flavon. Moreover, upon $SU(4) \otimes SU(2)_L \otimes SU(2)_R \rightarrow SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ it can split only into a pair of states $\chi_2^{q,l}$ giving rise to universal entries in the quark and lepton sectors respectively. However, in our setup this splitting can not emerge at the lowest level and thus the $b - \tau$ Yukawa unification is preserved up to higher order corrections.

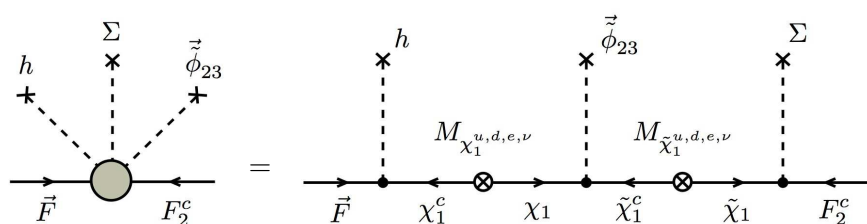


Figure 2: The structure of the Georgi-Jarlskog operator leading to the desired Clebsch-Gordon coefficient in the charged lepton sector giving the proper m_s/m_μ ratio. Since the projection of $\langle \Sigma \rangle$ in the $Y = 0$ direction is zero, the tri-bimaximal mixing in the neutrino sector remains unaffected.

The structure of the higher order operators responsible for the Georgi-Jarlskog structure (requiring an extra type-1 messenger field denoted by $\tilde{\chi}_1$) and the 1-3 and 2-3 CKM mixings is shown in figures 2 and 3.

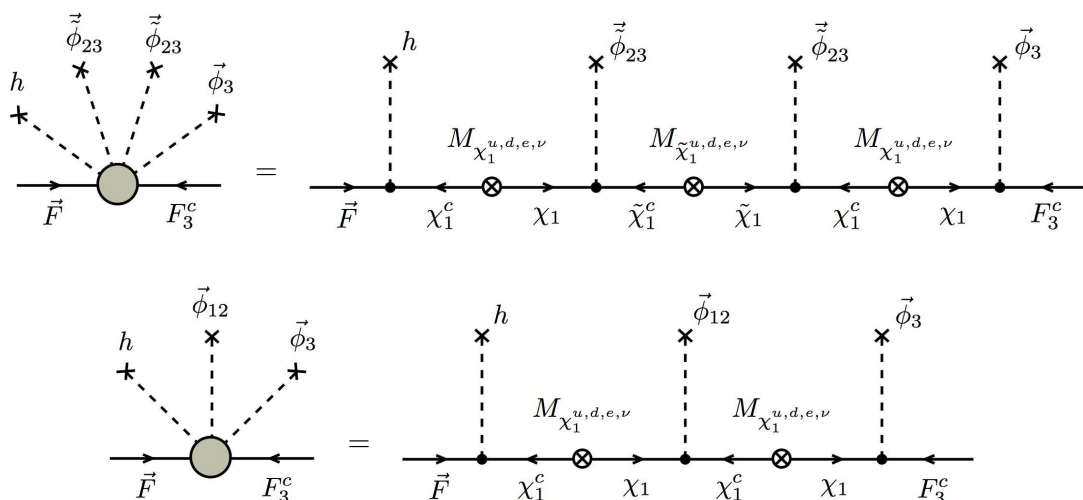


Figure 3: The leading contributions to the Yukawa 13 and 23 entries emerge automatically due to the particular χ_1 and $\vec{\phi}_{23}$ quantum numbers. The $SO(3)$ indices are contracted so that the leftmost $\vec{\phi}_{23}$ couples to \vec{F} and the $\vec{\phi}_{23}$ flavon on the right saturates the $SO(3)$ index of $\vec{\phi}_3$. This is the only option since $\tilde{\chi}_1$ is an $SO(3)$ singlet.

3.2 The Yukawa matrices

With the messenger sector specified, we can now return to the Dirac Yukawa matrix structure in eq. (2.6), and express the separate Yukawa matrices in each charge sector in terms of the messenger masses. The relevant expressions read (dropping the LR subscript for simplicity):

$$Y^u = \begin{pmatrix} 0 & y_{123}\varepsilon_{123}^u & y_{12}\varepsilon_{12}^u\varepsilon_3^u \\ y_{23}\varepsilon_{23}^u & y_{123}\varepsilon_{123}^u - 2y_{GJ}\tilde{\varepsilon}_{23}^u\sigma^u & \tilde{y}_{23}(\tilde{\varepsilon}_{23}^u)^2\varepsilon_3^u \\ -y_{23}\varepsilon_{23}^u & y_{123}\varepsilon_{123}^u - 2y_{GJ}\tilde{\varepsilon}_{23}^u\sigma^u & y_3'\varepsilon_3^u + y_3\varepsilon_3^u \end{pmatrix} \quad (3.2)$$

$$Y^d = \begin{pmatrix} 0 & y_{123}\varepsilon_{123}^d & y_{12}\varepsilon_{12}^d\varepsilon_3^d \\ y_{23}\varepsilon_{23}^d & y_{123}\varepsilon_{123}^d + y_{GJ}\tilde{\varepsilon}_{23}^d\sigma^d & \tilde{y}_{23}(\tilde{\varepsilon}_{23}^d)^2\varepsilon_3^d \\ -y_{23}\varepsilon_{23}^d & y_{123}\varepsilon_{123}^d + y_{GJ}\tilde{\varepsilon}_{23}^d\sigma^d & y_3'\varepsilon_3^d + y_3\varepsilon_3^d \end{pmatrix} \quad (3.3)$$

$$Y^e = \begin{pmatrix} 0 & y_{123}\varepsilon_{123}^e & y_{12}\varepsilon_{12}^e\varepsilon_3^e \\ y_{23}\varepsilon_{23}^e & y_{123}\varepsilon_{123}^e + 3y_{GJ}\tilde{\varepsilon}_{23}^e\sigma^e & \tilde{y}_{23}(\tilde{\varepsilon}_{23}^e)^2\varepsilon_3^e \\ -y_{23}\varepsilon_{23}^e & y_{123}\varepsilon_{123}^e + 3y_{GJ}\tilde{\varepsilon}_{23}^e\sigma^e & y_3'\varepsilon_3^e + y_3\varepsilon_3^e \end{pmatrix} \quad (3.4)$$

$$Y^\nu = \begin{pmatrix} 0 & y_{123}\varepsilon_{123}^\nu & y_{12}\varepsilon_{12}^\nu\varepsilon_3^\nu \\ y_{23}\varepsilon_{23}^\nu & y_{123}\varepsilon_{123}^\nu & \tilde{y}_{23}(\tilde{\varepsilon}_{23}^\nu)^2\varepsilon_3^\nu \\ -y_{23}\varepsilon_{23}^\nu & y_{123}\varepsilon_{123}^\nu & y_3'\varepsilon_3^\nu + y_3\varepsilon_3^\nu \end{pmatrix} \quad (3.5)$$

where we have used the following abbreviations (f stands for u, d, ν and e):

$$\varepsilon_{23}^f \equiv \frac{|\langle \vec{\phi}_{23} \rangle|}{M_{\chi_1^f}}, \quad \varepsilon_{123}^f \equiv \frac{|\langle \vec{\phi}_{123} \rangle|}{M_{\chi_1^f}}, \quad \varepsilon_3^f \equiv \frac{|\langle \vec{\phi}_3 \rangle|}{M_{\chi_1^f}}, \quad \varepsilon_{12}^f \equiv \frac{|\langle \vec{\phi}_{12} \rangle|}{M_{\chi_1^f}}, \quad \tilde{\varepsilon}_{23}^f \equiv \frac{|\langle \vec{\phi}_{23} \rangle|}{M_{\chi_1^f}} \quad (3.6)$$

$$\varepsilon_3^q \equiv \frac{|\langle \vec{\phi}_3 \rangle|}{M_{\chi_2^q}}, \quad \varepsilon_3^l \equiv \frac{|\langle \vec{\phi}_3 \rangle|}{M_{\chi_2^l}} \quad \text{and} \quad \sigma^f \equiv \frac{|\langle \Sigma \rangle|}{M_{\chi_1^f}} \quad (3.7)$$

The above effective Yukawa matrices are obtained upon integrating out the heavy messenger sector, leading to the following relations between the dimensionless couplings:

$$y_{23} \equiv y_{\chi_1^c F h} y_{\chi_1 F_1^c \phi_{23}}, \quad y_{123} \equiv y_{\chi_1^c F h} y_{\chi_1 F_2^c \phi_{123}}, \quad y_3 \equiv y_{\chi_1^c F h} y_{\chi_1 F_3^c \phi_3}, \quad y_3' \equiv y_{\chi_2 F_3^c h} y_{\chi_2^c F \phi_3}, \quad (3.8)$$

$$y_{GJ} \equiv y_{\chi_1^c F h} y_{\tilde{\chi}_1 \chi_1^c \tilde{\phi}_{23}} y_{\tilde{\chi}_1 F_2^c \Sigma}, \quad \tilde{y}_{23} \equiv y_{\chi_1^c F h} y_{\tilde{\chi}_1 \chi_1^c \tilde{\phi}_{23}}^2 y_{\chi_1 F_3^c \phi_3}, \quad y_{12} \equiv y_{\chi_1^c F h} y_{\chi_1 \chi_1^c \phi_{12}} y_{\chi_1 F_3^c \phi_3}$$

3.3 The messenger masses

It is known that the quark masses and mixing angles are well described by the following textures [70]:

$$|Y^u| \sim \begin{pmatrix} 0 & \varepsilon^3 & O(\varepsilon^3) \\ \varepsilon^3 & \varepsilon^2 & O(\varepsilon^2) \\ O(\varepsilon^3) & O(\varepsilon^2) & 1 \end{pmatrix}, \quad |Y^d| \sim \begin{pmatrix} 0 & 1.5\bar{\varepsilon}^3 & 0.4\bar{\varepsilon}^3 \\ 1.5\bar{\varepsilon}^3 & \bar{\varepsilon}^2 & 1.3\bar{\varepsilon}^2 \\ O(\bar{\varepsilon}^3) & O(\bar{\varepsilon}^2) & 1 \end{pmatrix}, \quad (3.9)$$

with $\varepsilon \sim 0.05$ and $\bar{\varepsilon} \sim 0.15$. The charged lepton Yukawa matrix receives a form similar to the down quark Yukawa matrix, but with a ‘‘Georgi-Jarlskog’’ factor of 3 in the (2, 2) entry of the charged lepton matrix.

There is clearly a need to generate a sizeable splitting in the spectrum of the χ_1 -type messengers, in particular among the components coupled to the up and down matter sectors. If we intend to reproduce the hierarchies suggested by textures (3.9) the ratio of the down and up χ_1 -type messenger masses $M_{\chi_1^d}/M_{\chi_1^u} \equiv r$ should be of the order of $\bar{\varepsilon}^3/\varepsilon^3 \sim \mathcal{O}(10^{-1})$. Thus, we should make $M_{\chi_1^u}$ much heavier than $M_{\chi_1^d}$. Clearly, this is possible only if the common singlet masses in the superpotential do not dominate the χ_1 mass formula, otherwise we get always $r \rightarrow 1$. It is also insufficient to split the χ_1 by means of Clebsch-Gordon coefficients of an extra Σ -like Higgs field (in analogy with the Georgi-Jarlskog mechanism) because $|r|$ in such a case is confined between the minimum and maximum ratio of the relevant Clebsch-Gordon coefficients ($\mathcal{O}(1)$ numbers). Thus, we need an alternative mechanism giving mass to χ_1^u only without touching χ_1^d .

This goal can be most economically achieved assuming that the underlying dynamics of the χ_1 field is governed by interactions with an extra messenger X that can propagate the information about the Pati-Salam breaking (triggered by the VEVs of H -fields) to the up-sector only. Indeed, this is possible if the $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$ quantum numbers of X are chosen as $(15, 1, 1)$. In such a case the structure of the $SU(2)_R$ contraction in the relevant operator $\chi_1^c X \langle H \rangle$ picks up the component of H with nonzero VEV together with the up-type part of χ_1^c (i.e. X corresponds to the diagonal matrix in the $2 \otimes 2$ product of $SU(2)_R$ and thus must be identified with the singlet in $2 \otimes 2 = 3 \oplus 1$). The choice of 15 out of $4 \otimes \bar{4} = 15 \oplus 1$ of $SU(4)_C$ is then justified by the need to propagate the VEV not only to the neutrino-like component $\chi_1^{\nu c}$ (as the singlet would obviously do) but also to χ_1^{uc} . This is illustrated in figure 4.

$$\begin{aligned}
 & \begin{array}{c} M_{\chi_1^{u,d,e,\nu}} \\ \leftarrow \otimes \rightarrow \\ \chi_1^c \quad \chi_1 \end{array} = \begin{array}{c} M_{\chi_1} \\ \leftarrow \times \rightarrow \\ \chi_1^c \quad \chi_1 \end{array} + \begin{array}{c} \begin{array}{c} H \\ \times \\ \vdots \\ \bullet \end{array} \leftarrow \begin{array}{c} \rightarrow \\ \times \\ \rightarrow \end{array} \begin{array}{c} \bullet \\ \vdots \\ \times \\ \bar{H} \end{array} \rightarrow \\ \chi_1^c \quad X \quad X^c \quad \chi_1 \end{array} + \dots \\
 \\
 & \begin{array}{c} M_{\chi_2^{q,l}} \\ \leftarrow \otimes \rightarrow \\ \chi_2^c \quad \chi_2 \end{array} = \begin{array}{c} M_{\chi_2} \\ \leftarrow \times \rightarrow \\ \chi_2^c \quad \chi_2 \end{array} + \begin{array}{c} \begin{array}{c} H \\ \times \\ \vdots \\ \bullet \end{array} \leftarrow \begin{array}{c} \rightarrow \\ \times \\ \rightarrow \end{array} \begin{array}{c} \bullet \\ \vdots \\ \times \\ \bar{H} \end{array} \rightarrow \\ \chi_2^c \quad Y \quad Y^c \quad \chi_2 \end{array} + \dots \\
 \\
 & \begin{array}{c} M_{\tilde{\chi}_1^{u,d,e,\nu}} \\ \leftarrow \otimes \rightarrow \\ \tilde{\chi}_1^c \quad \tilde{\chi}_1 \end{array} = \begin{array}{c} M_{\tilde{\chi}_1} \\ \leftarrow \times \rightarrow \\ \tilde{\chi}_1^c \quad \tilde{\chi}_1 \end{array} + \dots
 \end{aligned}$$

Figure 4: The large mass splitting in the χ_1 multiplet is achieved by means of a relatively suppressed singlet mass term accompanied by a particular set of “level-2” messengers giving rise to a mass term for $\chi_1^{u,\nu}$ only. Since there is no need to split the masses of the $\tilde{\chi}_1$ multiplet the leading contribution to its mass can be dominated by the singlet mass term. In principle, one can try to generate a mass splitting in the χ_2 multiplet on similar grounds as for χ_1 ; however, at the leading level the gauge structure admits only “internal” contractions of the H and \bar{H} $SU(2)_R$ indices (and similarly, the $SU(2)_L$ indices of χ_2 and χ_2^c must saturate each other) and thus, regardless the quantum numbers of Y the splitting in the χ_2 sector is in general much milder.

Notice that this mechanism is not applicable to the χ_2 case as these messengers are $SU(2)_R$ singlets and the $SU(2)_L$ symmetry prevents the up and down components from splitting up to the electroweak scale. This is welcome as the exchange of χ_2 is assumed to be the source of the universal 33 entries in the Yukawa sector. On the other hand, one can split the masses of its quark-like and lepton-like components χ_2^q and χ_2^l , but this requires a messenger (Y) carrying both $SU(2)_L$ as well as $SU(2)_R$ doublet indices. Moreover, to get m_τ slightly bigger than m_b this calls for $M_{\chi_2^l} < M_{\chi_2^q}$ which can be achieved only if Y is not an $SU(4)_C$ singlet (otherwise the lepton part is picked up by the VEVs of H and \bar{H}). Thus, to achieve such a splitting at the lowest effective operator order one must make use of higher Pati-Salam representations that seem disfavoured by strings. Thus, sticking to small multiplets such an effect is expected to arise from higher order operators justifying the mildness of the GUT-scale $b - \tau$ mass splitting.

Last note concerns the splitting in the $\tilde{\chi}_1$ messenger multiplet. We shall see in section 4 that the numerical fit is perfectly compatible with $M_{\tilde{\chi}_1^u} \sim M_{\tilde{\chi}_1^d} \sim M_{\tilde{\chi}_1^e}$ and thus there is no

field	$SU(4) \otimes SU(2)_L \otimes SU(2)_R$	$SO(3)$	$U(1)$	Z_2
X, X^c	$(15, 1, 1)$	3	∓ 3	+
Y, Y^c	$(15, 2, 2)$	1	0	-

Table 3: A sample “level-2” messenger sector giving rise to the desired level-1 Dirac sector messenger mass splitting $M_{\chi_1^u} \gg M_{\chi_1^d}, M_{\chi_2^l} \gtrsim M_{\chi_2^g}$.

need to generate a mass splitting within this sector. In other words, we assume the masses of $\tilde{\chi}_1$ components to be dominated by the explicit mass term in the superpotential.

3.4 The Majorana messenger sector

Concerning the Majorana mass terms given in eqs. (2.8)–(2.9), we do not enter a full analysis of the messenger sector here. The reason is that in the Majorana case there is no need to adjust the messenger masses in any particular way like in the Dirac sector, it is just enough to ensure a mild hierarchy of the light Majorana masses, that could be obtained in many different ways (c.f. the “richness” of the set of subleading operators given by formula (2.9)).

We therefore restrict our discussion here to the universal leading operators governing the diagonal entries of the mass matrix⁷ (2.10). Their Froggatt-Nielsen structure is depicted in figure 5. We employ a pair of extra messengers ψ and Ψ with the quantum numbers given in table 2. Notice that the structure of these graphs is such that the Dirac sector hierarchy associated with the $\vec{\phi}_{23}$ and $\vec{\phi}_{123}$ VEVs is effectively cancelled in the seesaw formula by the Majorana masses, and thus the light neutrino mass splitting is given by $m_2 \sim m_3$ at lowest order. This cancellation which was anticipated in section 2.4 was based on the assumption of equal expansion parameters in the Dirac and Majorana sectors. This assumption has now been verified, since we have seen here that the explicit messenger sector is common to both the Dirac and Majorana sector. Moreover, due to the presence of the heavy χ_1^ν messenger obeying $M_{\chi_1^\nu} \sim M_{\chi_1^u} \gg M_{\chi_1^{d,e}}$ there is a further suppression in the effective values of the neutrino ε_x^ν factors as in the up-sector case, $\varepsilon_x^{\nu,u} \sim r\varepsilon_x^{e,d}$.

To ensure the proper splitting between the first and second right-handed neutrino masses one can employ extra messenger fields to construct similar diagrams for the other terms allowed by symmetries, c.f. eq. (2.9).

4. Numerical analysis: a sample χ^2 fit

At this point, all the ingredients are fully specified and we can approach the fit of the quark masses and CKM mixing parameters, using the charged lepton masses as inputs.⁸

⁷The choice of quantum numbers of the messenger ψ is driven by simplicity, i.e. the need to pick up symmetric combinations from $4 \otimes 4$ of $SU(4)_C$ and $2 \otimes 2$ of $SU(2)_R$. Otherwise, the vertex with a pair of identical χ_1^ν vanishes.

⁸As we have already shown the neutrino sector automatically leads to a tri-bimaximal mixing so we do not include the lepton mixing parameters into the χ^2 analysis.

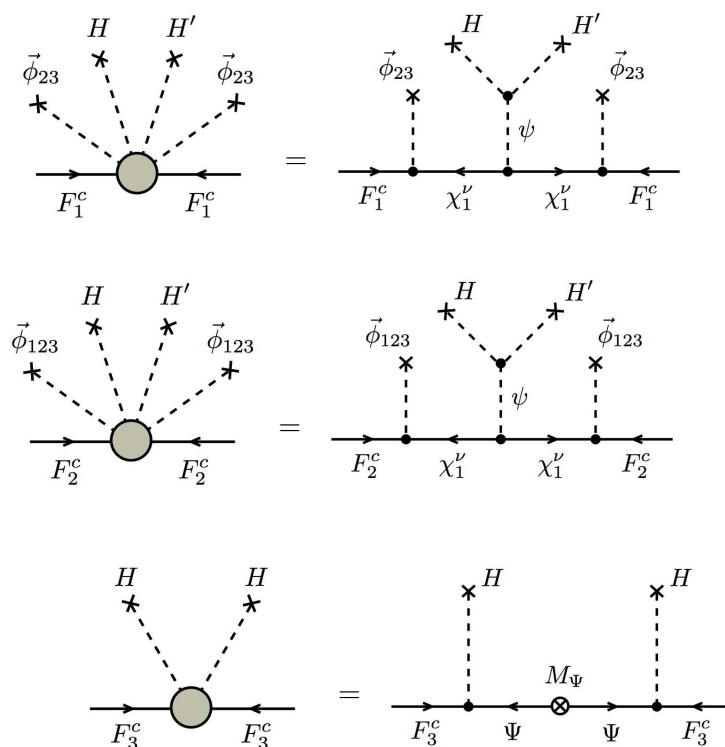


Figure 5: The basic lowest level (in number of extra suppressions carried by the VEVs of ϕ_{23} , ϕ_{123}, \dots flavons) effective operators contributing to the Majorana mass matrix in the neutrino sector. Note that there is no “mixed” term built up from the “halves” of the first and second graphs as the VEVs of ϕ_{23} and ϕ_{123} are orthogonal.

For the purposes of the numerical fit, it is convenient to introduce the parameters $y'_3 \varepsilon_3^q = z$, $a\delta z \equiv y_{23} \varepsilon_{23}^d$, $b\delta z \equiv y_{123} \varepsilon_{123}^d$, $cz \equiv y_{GJ} \tilde{\varepsilon}_{23}^d \sigma^d$, $dz \equiv y_3 \varepsilon_3^d$, $ez \equiv \tilde{y}_{23} (\tilde{\varepsilon}_{23}^d)^2 \varepsilon_3^d$ and $f\delta z \equiv y_{12} \varepsilon_{12}^d \varepsilon_3^d$. In terms of these parameters the charged sector Yukawa matrices in eqs. (3.2)–(3.4) can be expressed as:

$$Y^d = \begin{pmatrix} 0 & b\delta & f\delta \\ a\delta & b\delta + c & e \\ -a\delta & b\delta + c & d + 1 \end{pmatrix} z \quad (4.1)$$

$$Y^u = \begin{pmatrix} 0 & b\delta r & f\delta r^2 \\ a\delta r & b\delta r - 2cr\tilde{r} & er^2\tilde{r} \\ -a\delta r & b\delta r - 2cr\tilde{r} & dr + 1 \end{pmatrix} z \quad (4.2)$$

$$Y^e = \begin{pmatrix} 0 & b\delta s & f\delta s^2 \\ a\delta s & b\delta s + 3cs\tilde{s} & es^2\tilde{s} \\ -a\delta s & b\delta s + 3cs\tilde{s} & ds + t \end{pmatrix} z \quad (4.3)$$

where

$$r \equiv \frac{M_{\chi_1^d}}{M_{\chi_1^u}}, \quad \tilde{r} \equiv \frac{M_{\tilde{\chi}_1^d}}{M_{\tilde{\chi}_1^u}}, \quad s \equiv \frac{M_{\chi_1^d}}{M_{\chi_1^e}}, \quad \tilde{s} \equiv \frac{M_{\tilde{\chi}_1^d}}{M_{\tilde{\chi}_1^e}}, \quad t \equiv \frac{M_{\chi_2^q}}{M_{\chi_2^l}}. \quad (4.4)$$

The properly normalized mass matrices are then given by

$$M_u = \frac{1}{\sqrt{2}} Y^u v \sin \beta \equiv (Y^u/z)m, \quad M_{d,l} = \frac{1}{\sqrt{2}} Y^{d,l} v \cos \beta \equiv (Y^{d,l}/z) \frac{m}{\tan \beta} \quad (4.5)$$

where $\sqrt{2}m \equiv v_u(M_G)z$ stands for the single weak-scale dimensionfull parameter entering the numerical procedure. This parametrization is particularly convenient because of the possibility to get the optimum value of m (or equivalently z) analytically.

Regarding the number of input parameters used in the fit, we see from eqs. (4.1)-(4.3) that there are a total of 17 as follows: 11 from the 6 complex numbers⁹ a, b, c, d, e, f plus 6 real numbers $\delta, z, r, s, t, \tan \beta(M_G)$. This is generically¹⁰ the same as the number of quark and lepton sector observables (17 - the charged sector masses, CKM mixing angles and phase, $\tan \beta(M_Z)$ plus the lepton sector mixing angles). It is important to note that a “natural” fit (with all the dimensionless quantities having values around 1) is highly nontrivial, and the fact that we achieve this can be regarded as a success of the model.

We now consider a numerical determination of the values of the parameters defined above, subject to the following theoretical requirements. First we require all the unsuppressed parameters (i.e. all up to $\varepsilon_{23}^d, \varepsilon_{123}^d$ and ε_{12}^d) to fall in the natural $\mathcal{O}(1)$ domain. Next, the fit is obtained under the assumption that *there is only a single parameter δ governing all the ad-hoc suppressed quantities $\varepsilon_{23}^d, \varepsilon_{123}^d$ and ε_{12}^d* , i.e. $\varepsilon_{23,123,12}^d \propto \mathcal{O}(1) \times \delta$. In accord with the messenger sector dynamics we let r (c.f. section 3.3) depart from 1 (we have already seen that its preferred value in order to reproduce the textures (3.9) is in the $r \sim \mathcal{O}(10^{-1})$ range) but keep all the other messenger mass ratios around 1.

To obtain the desired Yukawa pattern as in eq. (3.9) (assuming $y_3^q \varepsilon_3^q = z \sim 1$ and keeping all the Yukawa couplings at $\mathcal{O}(1)$ level) one can estimate the magnitude of the extra suppression factors $\delta_{23,123}$ in ε_{23}^d and ε_{123}^d to be about $\varepsilon_{23}^d \sim \varepsilon_{123}^d \sim 0.003$. To make this suppression potentially natural (i.e. universal), also the small extra factor in the VEV of ϕ_{12} should have roughly the same value and we should assume $\varepsilon_{12}^d \sim 0.003$ as well. Is this compatible with the physical value of the 1-3 CKM mixing angle? Notice that with $r \ll 1$ the 13 term in the up Yukawa matrix is strongly suppressed and θ_{13}^q comes entirely from the down sector. Thus, $\theta_{13}^q \sim 0.003$ is obtained in a completely natural way provided $|y_{12}\varepsilon_3^d| \sim 1$, that corresponds to our assumption of no extra suppression factor in the ϕ_3 VEV.

The results of the numerical fit, subject to the above theoretical requirements, are displayed in tables 4 and 5. In performing this we took the input data at the GUT scale from [71] (slightly updated in view of some of the recent light quark sector changes indicated in [72]). The input data depend strongly on the value of $\tan \beta$ and since the present setup strongly favours the large $\tan \beta$ regime we take $\tan \beta(M_Z) = 55$. It is well known that in such a case there are potentially large radiative corrections to the down-type quark

⁹As a matter of fact, the phase of a does not matter at the leading order, see also the note in the caption of table 4.

¹⁰In principle the number of input parameters could be reduced further, for example due to the proximity of s to 1 in table 4 in the “case 2” fit corresponding to the common dynamical origin of the masses of χ_1^d and χ_1^e .

parameters		values (case 1)		values (case 2)	
δ		2.910×10^{-3}		3.153×10^{-3}	
$ a $	ϕ_a	3.805	0	1.17	π
$ b $	ϕ_b	2.262	3.344	0.759	2.406
$ c $	ϕ_c	0.045	0.206	0.014	1.655
$ d $	ϕ_d	1.330	0.053	0.021	2.108
$ e $	ϕ_e	0.086	0.274	0.035	0.905
$ f $	ϕ_f	2.284	1.529	1.029	5.892
m (or z)		55.97 (or 0.672)		88.79 (or 1.066)	
r		0.040		0.087	
\tilde{r}		1		1	
s		0.696		1.013	
\tilde{s}		1		1	
t		0.971		0.977	
$\tan \beta(M_G)$		64.96		52.64	

Table 4: The sample sets of input parameters corresponding to the fits of the quark and lepton sector measurables given in table 5. ϕ_a, \dots, ϕ_f are the phases of the complex parameters a, \dots, f . Notice the irrelevance of the phase of a which affects only the first columns of the relevant Yukawa matrices and thus can be rotated away. Note also that the smallness of $|c|$ and $|e|$ is natural because of their higher order effective operator origin. Concerning the value of d in “case 2”, this can be natural provided all the χ_1 messengers are somewhat heavier than χ_2 and the would-be suppression in the other χ_1 -driven Yukawa entries is redistributed into δ , $\tilde{\chi}_1$ masses, flavon VEVs and the couplings in the underlying theory. In other words, small $|d|$ is an issue of reparametrization rather than naturalness.

masses, in particular m_b , which can reach up to order 50% or even more. Unfortunately, the analysis [71] does not take most of these important radiative corrections into account. Thus, a detailed analysis including these effects, which strongly depend on the precise details of the SUSY spectrum, would be required in order to provide a more accurate fit to the data. This is beyond the scope of the current paper.

It can be easily checked that a good fit of the quark and charged lepton data can be obtained for instance for the values of the relevant parameters given in table 4. These sample solutions give relatively good fits of all the quark and lepton sector observables (c.f. table 5) and lead to the following Yukawa hierarchies at the unification scale (for “case 1” and “case 2” parameter sets respectively):

$$|Y^d|/z \approx \begin{pmatrix} 0 & 0.0066 & 0.0066 \\ 0.0110 & 0.0382 & 0.0858 \\ 0.0110 & 0.0382 & 2.329 \end{pmatrix}, \begin{pmatrix} 0 & 0.0024 & 0.0032 \\ 0.0037 & 0.0162 & 0.0349 \\ 0.0037 & 0.0162 & 0.9896 \end{pmatrix} \quad (4.6)$$

Observables (evolved to M_G)		case 1		case 2	
quantity	input	fit	pull (σ)	fit	pull (σ)
m_u [MeV]	0.55 ± 0.25	1.67	+4.46	0.55	\times
m_c [MeV]	210_{-21}^{+15}	215	+0.28	212.0	+0.08
m_t [GeV]	95_{-21}^{+69}	58.9	-1.75	88.70	-0.31
m_d [MeV]	1.23 ± 0.41	1.66	+1.06	0.93	-0.73
m_s [MeV]	21.6 ± 5.2	33.7	+2.32	27.77	+1.19
m_b [GeV]	$1.42_{-0.19}^{+0.48}$	2.01	+1.23	1.67	+0.52
m_e [MeV]	0.357	0.357	\times	0.357	\times
m_μ [MeV]	75.29	75.29	\times	75.29	\times
m_τ [GeV]	1.63	1.63	\times	1.63	\times
$\sin \theta_{12}^{\text{CKM}}$	0.2243 ± 0.0016	0.2273	+1.93	0.2248	+0.33
$\sin \theta_{23}^{\text{CKM}}$	0.0351 ± 0.0013	0.0369	+1.38	0.0351	+0.01
$\sin \theta_{13}^{\text{CKM}}$	0.0032 ± 0.0005	0.0032	-0.04	0.0031	-0.08
δ_{CKM}	$60^\circ \pm 14^\circ$	59.6°	-0.03	59.7°	-0.02
$\tan \beta(M_G)$	$52.1_{-16.5}^{+4.3}$	64.9	+2.1	52.64	+0.13
Sample χ^2 fit		Total χ^2 :	41.09	Total χ^2 :	2.45

Table 5: The GUT-scale input values and a sample χ^2 -fit of the quark and charged lepton masses and the CKM mixing parameters. The inputs correspond to the large $\tan \beta(M_Z) = 55$ regime at the MSSM scale $\sim 1\text{TeV}$ (for further details see the notes in the text). In “case 1” the analysis is performed at the leading order in the number of flavon insertions, which leads to small deviations in the first generation masses governing the total χ^2 function. The second column (“case 2”) gives an example of a very good fit in case of a tiny subleading entries $\sim \mathcal{O}(10^{-5})$ potentially arising at next-to-leading order. Due to $M_{\chi_1^d} \sim M_{\chi_1^e}$ this solution also corresponds to a regime with a simplified messenger sector dynamics; for details see the text and table 4.

$$|Y^u|/z \approx \begin{pmatrix} 0 & 0.0002 & 0 \\ 0.0004 & 0.0038 & 0.0001 \\ 0.0004 & 0.0038 & 1.0528 \end{pmatrix}, \begin{pmatrix} 0 & 0.0002 & 0 \\ 0.0003 & 0.0023 & 0.0003 \\ 0.0003 & 0.0023 & 0.9991 \end{pmatrix} \quad (4.7)$$

$$|Y^e|/z \approx \begin{pmatrix} 0 & 0.0046 & 0.0032 \\ 0.0077 & 0.0889 & 0.0416 \\ 0.0077 & 0.0889 & 1.8960 \end{pmatrix}, \begin{pmatrix} 0 & 0.0024 & 0.0033 \\ 0.0038 & 0.0455 & 0.0358 \\ 0.0038 & 0.0455 & 0.9666 \end{pmatrix} \quad (4.8)$$

One can verify easily that the portion of the lepton mixing coming from the charged lepton Yukawa is negligible. Moreover, the Cabibbo mixing emerges predominantly from the down-type Yukawa à la Gatto et al. [73] which is welcome.

Remarkably enough, more than 60% of the total χ^2 of this sample fit comes from the light (u, d, s) masses that are quite sensitive to the subleading corrections. The potentially dramatic effect of these contributions can be appreciated in “case 2” fit. Indeed, even as tiny as order 10^{-5} corrections to the Yukawa matrices above allow for a dramatic improvement

of the χ^2 value, c.f. table 5, “case 2”.¹¹ Recall that such extra factors emerge in a natural way from higher order effective operators like for instance

$$W^{h.o.} = y_a \frac{1}{M^2} \vec{F}.(\vec{\phi}_{23} \times \vec{\phi}_{12}) F_1^c h + y_b \frac{1}{M^3} \vec{F}.(\vec{\phi}_{23} \times \vec{\phi}_{12}) F_2^c \Sigma h + \dots \quad (4.9)$$

Moreover, since ϕ_{12} is a $U(1) \otimes Z_2$ singlet such terms emerge in a natural way upon inserting the ϕ_{12} VEV into the existing operators (2.1) without any need to enlarge the messenger sector.

Concerning the qualitative features of the fits, for “case 1” there seems to be preference of s slightly below 1 leading to m_s in the upper σ region. This originates from the fact that the optimal $M_{\chi_1^d}/M_{\chi_1^u}$ ratio ~ 0.04 leads to a natural value for the double ratio of the physical masses

$$x^{\text{fit}} = \frac{m_c/m_t}{m_s/m_b} \sim 2r \sim 0.08 \quad (4.10)$$

and thus m_s should be in its upper- σ region to lower the central value $x^c \sim 0.116$. Consequently, the pure Georgi-Jarlskog relation $m_s/m_\mu = 1/3$ (perfectly valid for m_s around 25 MeV at the GUT scale) is slightly violated. Note that since s parametrizes the mass splitting of the type-1 messengers, their masses are anyway expected to differ after the Pati-Salam symmetry breaking and there is no technical problem to receive s around 0.8 as suggested by the numerics.

In “case 2” solution there is no such need and all the messengers up to those coming from the χ_1 multiplet are nearly degenerate, that suggests further simplification of the underlying theory and, as a consequence, higher level of predictivity of the model. Moreover, the charged lepton masses and mixing angles conform the Georgi-Jarlskog scenario and thus for instance $\theta_{12}^e \approx Y_{12}^e/Y_{22}^e \approx Y_{12}^d/3Y_{22}^d \approx \theta_{12}^d/3 \approx 0.05$. This means that the charged lepton corrections to tri-bimaximal neutrino mixing are closely related to the Cabibbo angle, in particular [63, 64]:

$$\theta_{13} \approx \frac{1}{\sqrt{2}} \theta_{12}^e \approx 2^\circ \quad (4.11)$$

$$\theta_{12} + \theta_{13} \cos(\delta_{MNS} - \pi) \approx 35.26^\circ \quad (4.12)$$

where eq. (4.12) is the sum rule [63, 64] with δ_{MNS} being the MNS CP phase which enters neutrino oscillations, and 35.26° follows from tri-bimaximal neutrino mixing.

With the fit parameters in hand, we see that the consistency of the model requires the following hierarchy of the various mass scales present:

$$M_{X,Y} > M_{GUT} > M_{\chi_1^u} \gg M_{\tilde{\chi}_1} \gtrsim M_{\chi_1^d} \gtrsim M_{\chi_2^g} \sim M_{\chi_2^l} > \langle \Sigma \rangle, \langle \phi_3 \rangle, \langle \tilde{\phi}_{23} \rangle \gg \delta \langle \phi_{23} \rangle, \delta \langle \phi_{123} \rangle, \delta \langle \phi_{12} \rangle. \quad (4.13)$$

¹¹In the present case (“case 2” in table 5) we allowed for extra variations of order 10^{-5} in the Yukawa entries under consideration that is completely harmless for the heavy masses and CKM mixing parameters but can have dramatic effects on the “small” eigenvalues of $Y_{u,d,l}$, in particular the up-quark Yukawa, that is (due to the large hierarchy in the up-sector) by far the smallest eigenvalue in the game and as such also the most sensitive to the subleading corrections.

The first two inequalities satisfy the need to generate the proper effective masses of the Dirac sector messenger fields dynamically via exchange of the “level-2” messengers X and Y . The third, fourth, fifth and sixth relation ensures the proper Dirac sector messenger hierarchies described above. The last two relations justify the expansions in the number of Σ and flavon insertions used throughout the analysis. As it was argued above there is no problem to satisfy these relations with all the relevant tree-level Yukawa couplings (c.f. eq. (3.8)) at the $\mathcal{O}(1)$ level. Thus, the presented fit is natural. Notice also that the mildly suppressed 22, 32 and 23 Yukawa entries are reasonable as they arise from higher order effective operators and thus for instance $\tilde{\varepsilon}_{23}^d \sim \mathcal{O}(10^{-1})$ as suggested by the fit. All parameters are therefore either of order unity, or their smallness is accounted for by a ratio of mass scales, apart from the so far unexplained smallness of the parameter δ . In the next section we shall interpret δ as a volume suppression factor emerging in a higher dimensional theory.

5. The 5d orbifold GUT model based on $SO(3) \times SO(10)$

5.1 Introduction

The ad-hoc suppression factors present in the 4d model we proposed in the first part of this paper receive a simple justification once the theory is promoted to more than 4 dimensions. Indeed, every field propagating in the higher-dimensional bulk receives (from the point of view of the 4d effective theory) a volume suppression factor that can be used to generate extra hierarchies in its couplings to the localized fields.

As we have shown in section 4, the physical observables can be fitted even under the nontrivial assumption that the extra suppression factors in the effective couplings of the VEVs of the ϕ_{23} , ϕ_{123} , ϕ_{12} flavons (and an extra Higgs pair) all coincide. Thus, it is natural to ask whether such an effective 4d model can be understood as a low-energy limit of a more fundamental 5d theory.

Moreover, unlike the 4d Pati-Salam model, such an embedding could be viewed as a “true” 5d grand unified model that (as an orbifold GUT) can naturally accommodate the incomplete GUT multiplets of the effective model provided they live on the orbifold fixed point with a reduced gauge symmetry.

5.2 The setup

We assume a variation of the “standard” 5-dimensional $SO(10)$ SUSY GUT compactified on the $S^1/Z_2 \times Z'_2$ orbifold [29, 35, 37, 38]. The first Z_2 orbifold projection acting on the fifth coordinate y as $y \rightarrow -y$ is used to reduce the 5-dimensional N=1 supersymmetry (equivalent to N=2 SUSY in 4 dimensions) to an effective 4-dimensional N=1 SUSY while the full $SO(10)$ gauge symmetry is reduced to the Pati-Salam $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ on the brane located at the fixed point (the Pati-Salam brane) of the second projection $Z'_2 : y \rightarrow \pi R - y$. In accordance to the 4d model we assume an $SO(3)$ flavour symmetry acting on the left-handed matter multiplets living as incomplete $SO(10)$ multiplets at the Pati-Salam brane. As before, the flavour symmetry is augmented by an additional global $U(1) \otimes Z_2$. The graphical representation of the setup is given in figure 6.

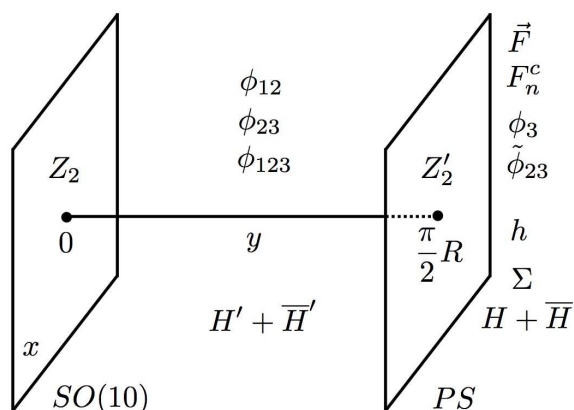


Figure 6: The physical 5-d setup: $S^1/Z_2 \times Z'_2$ orbifold is used to break the 5-dimensional N=1 SUSY in the bulk to the 4-d N=1 SUSY of the MSSM, and provides for the first step in the symmetry breaking of the unified SO(10) gauge group. The geometrical location of some of the flavon fields yields the necessary suppression factors in the relevant Yukawa textures.

The full set of brane and bulk fields used in our construction can be found in table 6. We take the advantage of the reduced gauge symmetry to put all the incomplete SO(10) multiplets to the PS brane. In order to get $\mathcal{O}(1)$ Yukawa couplings for the third generation fermions and only a mild suppression of the second generation masses the Georgi-Jarlskog Higgs field Σ as well as the light Higgs bidoublet live there¹² together with the flavons that should couple to the matter bilinears without further suppression factors (ϕ_3 and $\tilde{\phi}_{23}$).

To obtain a natural suppression of the first and the off-diagonal second family Yukawa couplings we put the flavons responsible for these interactions to the bulk, i.e. ϕ_{23} , ϕ_{123} and ϕ_{12} are assumed to propagate in 5 dimensions. Since there is a need to arrange an extra suppression factor in the first and second generation Majorana masses the relevant “primed” Higgs pair ($H' \oplus \bar{H}'$) also lives in the bulk.

As we shall see, such a setup upon compactification naturally leads to the effective Pati-Salam model constructed in previous sections with the desired Yukawa textures for the Dirac and Majorana mass matrices.

5.3 The 5d superpotential

The construction goes along similar lines as in the 4d case. In 5d, the relevant pieces of superpotential read:

$$\begin{aligned}
 W_Y^{\text{leading}} \propto & \int dy \delta\left(y - \frac{\pi}{2}R\right) \times \\
 & \times \frac{1}{M_m} \left(\frac{1}{\sqrt{M_*}} y_{23} \vec{F} \cdot \vec{\phi}_{23} F_1^c + \frac{1}{\sqrt{M_*}} y_{123} \vec{F} \cdot \vec{\phi}_{123} F_2^c + y_3 \vec{F} \cdot \vec{\phi}_3 F_3^c \right) h
 \end{aligned} \tag{5.1}$$

¹²Notice that there is no doublet-triplet splitting problem associated to the light doublets since they enter the game as an incomplete SO(10) multiplet.

$$\begin{aligned}
 W_Y^{subl.} &\propto \int dy \delta\left(y - \frac{\pi}{2}R\right) \times \\
 &\times \left[\frac{1}{M_m^3} \tilde{y}_{23} \vec{F} \cdot \vec{\phi}_{23} (\vec{\phi}_3 \cdot \vec{\phi}_{23}) F_3^c h + \right. \\
 &\quad \left. \frac{1}{M_m^2 \sqrt{M_*}} y_{12} \vec{F} \cdot (\vec{\phi}_3 \times \vec{\phi}_{12}) F_3^c h + \frac{1}{M_m^2} y_{GJ} \vec{F} \cdot \vec{\phi}_{23} F_2^c h \Sigma \right] + \dots
 \end{aligned} \tag{5.2}$$

$$\begin{aligned}
 W_M^{leading} &\propto \int dy \delta\left(y - \frac{\pi}{2}R\right) \times \\
 &\times \left[\frac{1}{M_m} w_3 F_3^{c2} H^2 + \frac{1}{M_*^{3/2} M_m^3} (w_1 F_1^{c2} H H' \phi_{23}^2 + w_2 F_2^{c2} H H' \phi_{123}^2) \right]
 \end{aligned} \tag{5.3}$$

$$\begin{aligned}
 W_M^{subl.} &\propto \int dy \delta\left(y - \frac{\pi}{2}R\right) \left[\frac{1}{M_m^4 M_*^2} w_4 F_1^{c2} H'^2 (\vec{\phi}_3 \times \vec{\phi}_{123}) \cdot \vec{\phi}_{23} + \right. \\
 &\quad + \frac{1}{M_m^4 M_*^{3/2}} w_5 F_2^{c2} H H' (\vec{\phi}_{23} \times \vec{\phi}_{12}) \cdot \vec{\phi}_{23} + \frac{1}{M_m^4 M_*^2} w_7 F_1^c F_2^c H H' (\vec{\phi}_{23} \times \vec{\phi}_{123}) \cdot \vec{\phi}_{12} + \\
 &\quad \left. + \frac{1}{M_m^4 M_*^{3/2}} w_9 F_1^c F_3^c H H' (\vec{\phi}_3 \times \vec{\phi}_{23}) \cdot \vec{\phi}_{12} \right] + \dots
 \end{aligned} \tag{5.4}$$

where as before M_m stands for the masses of the Froggatt-Nielsen messenger fields, R corresponds to the volume of the extra dimension and the dimensionfull parameter $\sqrt{M_*}$ is associated with every field propagating in the bulk to keep the action dimensionless. It is then easy to see that integrating over y and assigning

$$\delta \equiv \frac{1}{\sqrt{2\pi R M_*}} \tag{5.5}$$

we recover all the relevant 4d operators of section 2 with a natural bulk-suppression δ in all the factors including VEVs of $\phi_{23}, \phi_{123}, \phi_{12}$ and H' .

Thus, the 4d theory presented in the first part of the current paper can be viewed as a low-energy limit of a 5d orbifold GUT model.

6. Conclusions

The problem of fermion masses and mixings has become more interesting over recent years with the discovery of neutrino mass and mixings, which show that the neutrino sector differs markedly from the charged fermion sector. The most promising approaches to understanding fermion masses and mixings seem to involve a combination of GUT and Family Symmetries. The most recent neutrino oscillation data is consistent with tri-bimaximal mixing, which could naturally result from the see-saw mechanism with CSD where a non-Abelian Family Symmetry such as $SU(3)$, $SO(3)$, or one of its discrete subgroups, provides a framework for the necessary vacuum alignment of flavon VEVs [63, 65–67]. We have considered a specific 4d model based on Pati-Salam unification and $SO(3)$ gauged Family Symmetry, although it could be extended to the case where $SO(3)$ is replaced by one of its discrete subgroups such as A_4 , where the problem of vacuum alignment is potentially simpler [66–68]. In the relevant low energy effective Yukawa operators the $SO(3)$

field	Pati-Salam	$SO(3)$	$U(1)$	Z_2	location
\vec{F}	$(4, 2, 1)$	3	0	+	PS brane
F_1^c	$(\bar{4}, 1, 2)$	1	+2	-	PS brane
F_2^c	$(\bar{4}, 1, 2)$	1	+1	+	PS brane
F_3^c	$(\bar{4}, 1, 2)$	1	-3	-	PS brane
h	$(1, 2, 2)$	1	0	+	PS brane
H, \bar{H}	$(4, 1, 2), (\bar{4}, 1, 2)$	1	± 3	+	PS brane
H', \bar{H}'	$(\bar{4}, 1, 2), (\bar{4}, 2, 1), \dots$	1	∓ 3	+	bulk
Σ	$(15, 1, 3)$	1	-1	-	PS brane
$\vec{\phi}_3$	$(1, 1, 1)$	3	+3	-	PS brane
$\vec{\phi}_{23}$	$(1, 1, 1)$	3	0	-	PS brane
$\vec{\phi}_{23}$	$(1, 1, 1)$	3	-2	-	bulk
$\vec{\phi}_{123}$	$(1, 1, 1)$	3	-1	+	bulk
$\vec{\phi}_{12}$	$(1, 1, 1)$	3	0	+	bulk

Table 6: The matter, Higgs and flavon content of the model. Placing some of the flavon and Higgs fields into the bulk leads to an effective “screening” of their 4-d couplings to the Pati-Salam brane matter multiplets via the relevant volume suppression factors.

flavons enter at the simplest possible one-flavon level, unlike $SU(3)$ where the lowest order operators must involve at least two flavons.

The existing analyses of models of this kind have so far been performed at the 4d effective non-renormalizable operator level [63, 65–67]. We have gone beyond existing analyses by considering, as well as the 4d effective non-renormalizable operators, also the underlying renormalizable 4d model in terms of a high energy messenger sector. This represents an explicit ultraviolet completion of the model. The messenger sector allows for effectively different expansion parameters in the different charged sectors. We performed a numerical analysis which shows that the model provides an excellent fit to the charged fermion mass spectrum. The model also predicts approximate tri-bimaximal lepton mixing via CSD due to vacuum alignment of flavon VEVs, with calculable deviations described by the neutrino sum rule. The strong hierarchy in the charged fermion sector, explained in terms of a small flavon VEV, gets cancelled in the neutrino sector, via the see-saw mechanism with sequential dominance, leading to $m_2 \sim m_3$ for the lowest order neutrino masses, with the mild neutrino hierarchy $m_2/m_3 \sim 1/5$ produced by higher order corrections necessarily present in the model.

We have shown how the model can originate from a 5d orbifold GUT based on $SO(3) \times SO(10)$. From the perspective of orbifold GUTs this provides significant progress since such models have not so far provided a convincing explanation of fermion masses and mixings. The small flavon VEVs responsible for the fermion mass hierarchy, which were postulated in an *ad hoc* way from the 4d point of view, are seen to originate from bulk volume suppression in the 5d theory. The bulk suppressed VEVs reduce the need for very high dimensional operators, allowing a simpler operator structure which can be more readily understood at the renormalizable level in terms of an explicit messenger sector. The Pati-

Salam symmetry is also shown to arise from a broken $SO(10)$ GUT. This demonstrates that, in the framework of a higher dimensional theory, models based on the gauged Family Symmetry $SO(3)$, or one of its discrete subgroups, can be fully consistent with $SO(10)$ Grand Unification.

To summarize, the model presented here provides a good description of fermion masses and mixings, and a natural explanation of tri-bimaximal lepton mixing. The framework of non-Abelian Family Symmetry and GUTs, which has been used to account for tri-bimaximal lepton mixing via CSD and vacuum alignment, has here been combined with orbifold GUTs. The resulting synthesis allows the fermion mass hierarchy to be explained by a small bulk suppressed flavon VEV, which simplifies the Yukawa operator structure considerably, allowing the ultraviolet completion of the model in terms of a renormalizable messenger sector.

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